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A NEW VERSION OF BOUNDARY INTEGRAL EQUATIONS AND THEIR APPLICATION TO DYNAMIC THREE-DIMENSIONAL PROBLEMS OF THE THEORY OF ELASTICITY†

A. O. VATUL'YAN and V. M. SHAMSHIN

Rostov-on-Don

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A version of boundary integral equations of the first kind in dynamic problems of the theory of elasticity is proposed, based on an investigation of the analytic properties of the Fourier transformant of the displacement vector, rather than on fundamental solutions. A system of three boundary integral equations of the first kind with Fredholm kernels is constructed, and the equivalence of the initial boundary-value problem on the vibrations of a bounded region and the system of boundary integral equations obtained is investigated. A version of the numerical realization, which combines the ideas of the classical method of boundary elements and the Tikhonov regularization method, is proposed. The results of numerical experiments are given. © 1998 Elsevier Science Ltd. All rights reserved.

When investigating dynamic problems of the theory of elasticity, the classical version of the method of boundary integral equations, based on a knowledge of the fundamental solutions [1, 2], enables one to obtain a solution inside the region using Somigliana integral operators, the kernels of which are expressed in terms of the fundamental Kupradze matrix, and have singularities. In a number of cases (in problems of the anisotropic theory of elasticity, in axisymmetric problems of the isotropic theory of elasticity, and problems of electroelasticity), the fundamental solutions cannot be expressed in explicit form. Integral representations of the fundamental solutions for plane problems of the anisotropic theory of elasticity have been constructed [3, 4], but their further practical application (the realization of a version of the method of boundary elements) requires the evaluation of multiple integrals when setting up appropriate algebraic systems. An alternative formulation of boundary integral equations of the first kind was proposed in [5] in dynamic problems of the anisotropic theory of elasticity, which does not require a knowledge of the fundamental solutions and which is based solely on an analysis of the characteristic polynomial of the operator of the theory of elasticity, on the assumption that the characteristic surfaces do not intersect. Unfortunately, in the isotropic theory of elasticity two of the characteristic surfaces coincide, and the proposed approach requires a more careful examination. Note that the ideas related to the investigation of the analyticity of the Fourier transformation and the formulation of the boundary integral equations were apparently first described in [6] when obtaining boundary integral equations of the first kind in the problem of the vibrations of a half-space with a rough boundary.

1. FORMULATION OF THE BOUNDARY INTEGRAL EQUATIONS OF THE FIRST KIND IN THREE-DIMENSIONAL DYNAMIC PROBLEMS OF THE ISOTROPIC THEORY OF ELASTICITY

Consider the steady vibrations of an elastic solid, occupying a simply connected region $V \subset \mathbf{R}^3$, star-like with respect to a certain sphere, with a piecewise-smooth boundary S. The equations of the steady vibrations of an isotropic elastic solid of density ρ with frequency ω have the form [7]

$$\nabla \cdot \mathbf{T} + \rho\omega^2 \mathbf{u} = 0 \tag{1.1}$$

where $\mathbf{u}$ is the displacement vector, $\mathbf{T}$ is the Cauchy stress tensor, and ε is the linear strain tensor

$$\mathbf{T} = 2\mu\left[\frac{\nu}{1-2\nu}\nabla \cdot \mathbf{u}\mathbf{E} + \varepsilon\right], \quad \varepsilon = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^T] \tag{1.2}$$

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We will assume that the following boundary conditions are specified

$$\mathbf{u}|_{S_u} = \mathbf{f}, \quad \mathbf{t}_n|_{S_\sigma} = \mathbf{g}, \quad \mathbf{t}_n = \mathbf{n}\cdot\mathbf{T}, \quad S = S_u \cup S_\sigma \tag{1.3}$$

We will construct a system of boundary integral equations of the first kind based on ideas proposed earlier in [5]. We will assume that the boundary-value problem (1.1)–(1.3) has a solution $\mathbf{u} \in W_2^1(V)$ for sufficiently smooth vector functions $\mathbf{f}$ and $\mathbf{g}$. We carry out an integral Fourier transformation of Eq. (1.1) with parameter $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, for which we multiply Eq. (1.1) by $e^{i\alpha\cdot\mathbf{x}}$ and integrate it over the region V. Converting the volume integrals into surface integrals using the Gauss–Ostrogradskii theorem, we obtain

$$\beta\alpha\alpha\cdot\tilde{\mathbf{u}} + \tau\tilde{\mathbf{u}} = \mathbf{v}(\alpha), \quad \beta = \frac{\mu}{1-2\nu}, \quad \tau = (\mu\alpha\cdot\alpha - \rho\omega^2) \tag{1.4}$$

where

$$\mathbf{v}(\alpha) = \int_S \mathbf{t}_n e^{i\alpha\cdot\mathbf{x}}dS - 2\beta\nu i\alpha\int_S \mathbf{n}\cdot\mathbf{u}e^{i\alpha\cdot\mathbf{x}}dS - \mu i\alpha\cdot\int_S (\mathbf{nu}+\mathbf{un})e^{i\alpha\cdot\mathbf{x}}dS \tag{1.5}$$

$$\tilde{\mathbf{u}} = \int_V \mathbf{u}e^{i\alpha\cdot\mathbf{x}}dV$$

and $\mathbf{n}$ is the unit vector normal to the surface S. Expanding relation (1.4) in components of the vector $\tilde{\mathbf{u}}$, we obtain

$$\begin{gathered}\tilde{\mathbf{u}} = \Delta^{-1}\mathbf{R}\cdot\mathbf{v}, \quad \Delta = \tau(\tau + \beta\alpha^2), \quad \mathbf{R} = \|r_{mn}\| \\ r_{mm} = \tau + (\alpha^2 - \alpha_m^2), \quad r_{mn} = -\beta\alpha_m\alpha_n, \quad n \neq m\end{gathered} \tag{1.6}$$

We know that the Fourier transformation of the vector function $\mathbf{u} \in W_2^1(V)$, where V is a simply connected region, star-like with respect to a certain sphere, is an exponential-type integral function [8] by virtue of the limit

$$|\tilde{\mathbf{u}}(\alpha)| \leqslant C\exp\left[\max_{\mathbf{x}\in V}|\mathbf{x}|\cdot|\operatorname{Im}\alpha|\right]$$

However, in view of representation (1.6), the principal part of the Laurent series of vector functions $\tilde{\mathbf{u}}$ is not identically equal to zero, since the components of $\tilde{\mathbf{u}}$ have simple poles and zeros Δ, i.e. when the hodograph of the vector α lies on spheres

$$\alpha = k_1\eta \text{ and } \alpha = k_2\eta \tag{1.7}$$

where

$$k_1^2 = \frac{\omega^2}{c_1^2}, \quad k_2^2 = \frac{\omega^2}{c_2^2}, \quad c_1^2 = \frac{\lambda+2\mu}{\rho}, \quad c_2^2 = \frac{\mu}{\rho}$$

$$\eta = (\cos\theta\cos\psi, \cos\theta\sin\psi, \sin\theta), \quad |\eta| = 1, \quad 0 \leqslant \psi < 2\pi, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$$

and c_1 and c_2 are the velocities of propagation of longitudinal and transverse waves.

To eliminate the contradiction which arises it is necessary to require that the coefficients of the principal part of the Laurent series should vanish, i.e. the residues on these spheres. Since the poles are simple, it is sufficient to require that $\mathbf{R}\cdot\mathbf{v}$ should vanish when α takes the values (1.7).

Suppose $\alpha = k_1\eta$. Then, starting from relation (1.6), we obtain a linear system of equations in the components of the vector $\mathbf{v}$, whence we find the first resolvent, which relates the components of the stress and strain vectors on the boundary of the region

$$\eta_1\nu_1(k_1\eta) + \eta_2\nu_2(k_1\eta) + \eta_3\nu_3(k_1\eta) = 0 \tag{1.8}$$

Suppose how that $\alpha = k_2\eta$. Then, as previously, we find the first and third resolvents

$$\eta\psi_1(k_2\eta)-\eta_1\psi_3(k_2\eta)=0,\quad \eta\psi_2(k_2\eta)-\eta_2\psi_3(k_2\eta)=0 \tag{1.9}$$

Thus, problem (1.1)–(1.3) reduces to solving a system of resolvents (1.8) and (1.9), which can be written in the form

$$\int_S \mathbf{K}^\sigma(\mathbf{x},\eta)\cdot\mathbf{t}_n(\mathbf{x})dS+\int_S \mathbf{K}^u(\mathbf{x},\eta)\cdot\mathbf{u}(\mathbf{x})dS=0 \tag{1.10}$$

$$\mathbf{K}^u=\mu i\mathbf{s}\cdot\begin{Vmatrix} An_1+B\eta_1 & An_2+B\eta_2 & An_3+B\eta_3 \\ D\eta_1-B\eta_3 & D\eta_2 & B\eta_1+D\eta_3 \\ C\eta_1 & B\eta_3+C\eta_2 & C\eta_3-B\eta_2 \end{Vmatrix}$$

$$\mathbf{K}^\sigma=\mathbf{e}\cdot\begin{Vmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_3 & 0 & -\eta_1 \\ 0 & -\eta_3 & \eta_2 \end{Vmatrix}$$

$$\mathbf{s}=\operatorname{diag}(-2k_1e_1,k_2e_2,k_2e_2),\quad \mathbf{e}=\operatorname{diag}(e_1,e_2,e_2),\quad e_j=\exp[ik_j\eta\cdot\mathbf{x}]$$

$$A=\frac{\nu}{1-2\nu},\quad B=\mathbf{n}\cdot\eta,\quad C=\eta\times\mathbf{i}_1\cdot\mathbf{n},\quad D=\eta\times\mathbf{i}_2\cdot\mathbf{n}$$

Taking boundary conditions (1.3) into account, we obtain that the initial problem reduces to a system of integral equations of the first kind with Fredholm kernels $\mathbf{K}^\sigma(\mathbf{x},\eta)$, $\mathbf{K}^u(\mathbf{x},\eta)$ defined on a Cartesian product $S\times S_1$ (S_1 is the unit sphere $|\eta|=1$)

$$\int_{S_u} \mathbf{K}^\sigma(\mathbf{x},\eta)\cdot\mathbf{t}_n(\mathbf{x})dS+\int_{S_\sigma} \mathbf{K}^u(\mathbf{x},\eta)\cdot\mathbf{u}(\mathbf{x})dS=\mathbf{F}(\eta),\quad |\eta|=1 \tag{1.11}$$

where

$$-\mathbf{F}(\eta)=\int_{S_\sigma} \mathbf{K}^\sigma(\mathbf{x},\eta)\cdot\mathbf{g}(x)dS+\int_{S_u} \mathbf{K}^u(\mathbf{x},\eta)\cdot\mathbf{f}(\mathbf{x})dS$$

Relations (1.11) are a system of boundary integral equations which relate the unknown components of the stress vector $\mathbf{t}_n|_{Su}$ and the strain displacement $\mathbf{u}|_{S\sigma}$.

Note that the kernels $\mathbf{K}^\sigma(\mathbf{x},\eta)$, $\mathbf{K}^u(\mathbf{x},\eta)$ are infinitely differentiable with respect to the variable η_j ($j=1,2,3$), while if the surface S is a Lyapunov surface, i.e. a surface of the class $C^{(1,\zeta)}$, we have

$$\mathbf{K}^u(\mathbf{x},\eta)\in C^{(0,\zeta)}(S)\times C^\infty(S_1),\quad \mathbf{K}^\sigma(\mathbf{x},\eta)\in C^\infty(S)\times C^\infty(S_1),\quad 0<\zeta<1$$

We know that the procedure for inverting Fredholm operators of the first kind, generally speaking, is an ill-posed problem, in view of the unboundedness of the inverse operator to (1.11) [9]. This situation is typical for Fredholm operators of the first kind with an arbitrary right-hand side $\mathbf{F}(\eta)$ from a certain class of functions. However, in a specific case, $\mathbf{F}(\eta)$, according to (1.11), has an extremely specific form, and this fact enables us to judge whether the inverse operator is bounded, on the basis of the solvability of the initial problem in $W_2^1(V)$ and the theorem of equivalence.

Theorem. Suppose $V\subset\mathbf{R}^3$ is a simply connected region with a piecewise-smooth boundary S. Then boundary-value problem (1.1)–(1.3) is equivalent to the problem of finding the boundary values $\mathbf{t}_n|_{Su}$ from integral equations (1.11).

Proof. Note that the system of boundary integral equations (1.11) is a consequence of boundary-value problem (1.1)–(1.3) by construction. We will show that if $\mathbf{u}|_{S\sigma}$ and $\mathbf{t}_n|_{Su}$ are a solution of system (1.11), they can be extended inside the region V, and the equations of motion (1.1) and Hooke's law (1.2) will be satisfied inside V.

Suppose the vector functions $\mathbf{u}$ and $\mathbf{t}_n$ are a solution of boundary integral equations (1.11). We will introduce an integral vector function $\mathbf{v}$, as given by (1.5), and a vector function $\tilde{\mathbf{u}}(\alpha)$, as given by (1.6).

By virtue of the system of boundary integral equations (1.11), $\tilde{\mathbf{u}}(d)$ are exponential-type integral functions, and by virtue of the Paley–Wiener theorem [10] are a Fourier transformation with carrier in V. It can be shown that the vector function

$$\mathbf{u}(\mathbf{x}) = \frac{1}{(2\pi)^3} \int_{\mathbf{R}^3} \tilde{\mathbf{u}}(\alpha) e^{-i\alpha\cdot\mathbf{x}} d\alpha \tag{1.12}$$

satisfies Eqs (1.1) and (1.2) by construction, in which case (1.6) is satisfied. Note that the elements of the matrix $\mathbf{R}$ possess the property $\mathbf{R}(t\alpha, t\omega) = t^2\mathbf{R}(\alpha, \omega)$. We will show that for a vector function defined by (1.12) and (1.6), boundary conditions (1.3) will also be satisfied.

In fact, suppose $\mathbf{y} \in S_u$ is a regular point on the boundary. We will prove that

$$\lim_{\mathbf{x}\to\mathbf{y}\in S_u} \mathbf{u}(\mathbf{x}) = \mathbf{f}(\mathbf{y})$$

To do this we consider a sphere $V_\varepsilon(\mathbf{y})$ with centre at the point $\mathbf{y}$ and of radius $\varepsilon > 0$. We introduce the region $V_{1\varepsilon} = V \cup V_\varepsilon(\mathbf{y})$, denote its boundary by $S_{1\varepsilon}$ and write Eqs (1.5) for $S_{1\varepsilon}$. Then

$$\lim_{\mathbf{x}\to\mathbf{y}\in S_u} \mathbf{u}(\mathbf{x}) = \lim_{\varepsilon\to 0} \mathbf{u}_\varepsilon(\mathbf{y})$$

where $\mathbf{u}_\varepsilon(\mathbf{y})$ is defined by (1.12) and (1.6), but instead of $\mathbf{v}(\alpha)$ we need to take $\mathbf{v}_\varepsilon(\alpha)$. We will represent the integral over $S_{1\varepsilon}$ in the form of the sum of two integrals over S_ε and over $S_\varepsilon^+(\mathbf{y})$, where S_ε is the part of the surface S lying outside the sphere $V_\varepsilon(\mathbf{y})$, while $S_\varepsilon^+(\mathbf{y})$ is the part of the sphere, bounding $V_\varepsilon(\mathbf{y})$, lying outside V.

The first integral in the limit as $\varepsilon \to 0$ gives the integral in the sense of the Cauchy principal value $\mathbf{T}(\mathbf{y})$. In the second integral we change the variables

$$\mathbf{x} - \mathbf{y} = \varepsilon\eta, \quad \alpha = \varepsilon^{-1}\gamma, \quad |\eta| = 1$$

Further, we take into account the fact that

$$\Delta(\varepsilon^{-1}\gamma, \omega) = \varepsilon^{-4}[\mu^2|\gamma|^4 + o(\varepsilon^2)], \quad \mathbf{R}(\varepsilon^{-1}\gamma, \omega) = \varepsilon^{-2}[\mathbf{R}_0(\gamma) + o(\varepsilon^2)], \quad \varepsilon \to 0 \tag{1.13}$$

and $\mathbf{u}(\mathbf{y} + \varepsilon\eta) = \mathbf{u}(\mathbf{y}) + o(\varepsilon)$, where $\mathbf{R}_0(t\gamma) = t^2\mathbf{R}_0(\gamma)$. By evaluating the integral over $S_\varepsilon^+(\mathbf{y})$ and taking into account the fact that on this surface $\mathbf{n} = \eta$, we obtain

$$\mathbf{u}(\mathbf{y}) = \mathbf{T}(\mathbf{y}) + \mathbf{f}(\mathbf{y})/2 \tag{1.14}$$

Proceeding in a similar way for the case when $\mathbf{x} \notin V$, we obtain

$$0 = T(\mathbf{y}) - \mathbf{f}(\mathbf{y})/2 \tag{1.15}$$

Hence, by virtue of (1.14) and (1.15) the first of the limit relations (1.3) is proved.

To prove the second limit relation, we obtain from (1.2) the stress vector $\mathbf{t}_n = \mathbf{n}\cdot\mathbf{T}$, taking (1.12) into account. Then

$$\tilde{t}^{(n)}(\alpha) = \Delta^{-1}(\alpha, \omega)\mathbf{Q}(\alpha, \omega, \mathbf{n})\cdot\mathbf{v}(\alpha)$$

and the elements of the matrix $\mathbf{Q}(\alpha, \omega, \mathbf{n})$ depend linearly on $\mathbf{n}$ and possess the property

$$\mathbf{Q}(t\alpha, t\omega, \mathbf{n}) = t^3\mathbf{Q}(\alpha, \omega, \mathbf{n}), \quad t > 0$$

Suppose $\mathbf{y} \in S_\sigma$. The procedure for proving the limit relation as a whole is a repetition of the previous construction, taking into account the fact that

$$\mathbf{Q}(\varepsilon^{-1}\gamma, \omega, \mathbf{n}) = \varepsilon^{-3}[\mathbf{Q}_0(\gamma)\mathbf{n} + o(\varepsilon^2)], \quad \varepsilon \to 0$$

(the sole difference from the above case is the fact that the limit of the integral over S_ε is understood in the sense of a finite Hadamard value).

Notes. 1. These equations can be conveniently used instead of the classical equations when it is required to determine only the boundary values of the unknowns, i.e. for example, when finding contact stresses, or when determining the displacement field on the stress-free part of the surface. Moreover, these equations can also be used when both the displacements and the stress vector are known on one part of the surface but the boundary conditions on the other part are unknown. This system of boundary integral equations then also enables one to determine the unknown boundary conditions if the method of quasisolutions is employed [11].

2. When $\omega = 0$, the system of boundary integral equations (1.11) serves as the condition for the solvability of the corresponding static problem of the theory of elasticity.

2. AXIALLY SYMMETRIC DEFORMATION OF A SOLID OF REVOLUTION

In the case of the axially symmetric deformation of a solid of revolution, we can put $\eta_2 = 0$ in (1.8) and (1.9) and, changing to a polar system of coordinates, we can integrate over the coordinate φ. The kernels of integral equations (1.11) can then be expressed in terms of Bessel functions, and the integration is carried out over the generatrix of the solid of revolution L.

Suppose the region V occupied by the solid of revolution is given by the equation

$$r = \Phi(z), \ a \leqslant z \leqslant b, \ S = L \times [0; 2\pi)$$

Then the vector of the outward normal has the form

$$\mathbf{n} = n_r \mathbf{e}_r + n_z \mathbf{e}_z, \ n_r = \frac{1}{\sqrt{1+(\Phi')^2}}, \ n_z = \frac{-\Phi'}{\sqrt{1+(\Phi')^2}}$$

where $\mathbf{e}_r$, $\mathbf{e}_z$ are the basis vectors in a polar system of coordinates. After integrating over the coordinate φ from 0 to 2π, boundary integral equation (1.11) takes the form

$$\int_{L_u} \mathbf{K}^\sigma(r,z,\eta_1,\eta_3)\cdot \mathbf{t}_n(r,z)dL + \int_{L_\sigma} \mathbf{K}^u(r,z,\eta_1,\eta_3)\cdot \mathbf{u}(r,z)dL = \mathbf{F}(r,z) \tag{2.1}$$

where

$$-\mathbf{F}(r,z) = \int_{L_\sigma} \mathbf{K}^\sigma(r,z,\eta_1,\eta_3)\cdot \mathbf{g}(r,z)dL + \int_{L_u} \mathbf{K}^u(r,z,\eta_1,\eta_3)\cdot \mathbf{f}(r,z)dL$$

$$L = L_u \cup L_\sigma, \quad \eta_1^2 + \eta_3^2 = 1$$

Here L_u is the generatrix of the surface S_u and L_σ is the generatrix of the surface S_σ. The kernels of boundary integral equation (1.11) in the axisymmetric case can be expressed by the following formulae

$$\mathbf{K}^u = \mu \begin{Vmatrix} A_1(G_{01}-G_{21})+A_2G_{11} & 0 & A_3G_{01}+A_4G_{11} \\ B_1(G_{02}-G_{22})+B_2G_{02} & 0 & B_4G_{12}+B_5G_{02} \\ 0 & C_1G_{22}+C_2G_{12} & 0 \end{Vmatrix}$$

$$\mathbf{K}^\sigma = \begin{Vmatrix} i\eta_i G_{11} & 0 & \eta_3 G_{01} \\ i\eta_3 G_{12} & 0 & -\eta_1 G_{02} \\ 0 & i\eta_3 G_{12} & 0 \end{Vmatrix}$$

$$G_{ij} = J_i(k_j\eta_1 r), \ i = 0,1,2; \ j = 1,2$$

$$A_1 = ik_1n_r(A+\eta_1^2), \ A_2 = 2k_1\eta_1\eta_3 n_z, \ A_3 = 2ik_1n_z(A+\eta_3^2)$$

$$A_4 = 2k_1\eta_1\eta_3 n_r, \ B_1 = -ik_2\eta_3 n_r, \ B_2 = k_2n_z(\eta_3^2 - k_2\mu\eta_1^2)$$

$$B_4 = k_2n_r(\eta_3^2 - \eta_1^2), \ B_5 = -2k_2\eta_1\eta_3 n_z, \ C_1 = ik_2\eta_1\eta_3 n_r, \ C_2 = k_2\eta_3^2 n_z$$

Here the following vectors serve as the unknowns

$$\mathbf{t}_n(r,z)|_{S_u} = t_r\mathbf{e}_r + t_\varphi \mathbf{e}_\varphi + t_z\mathbf{e}_z, \ \mathbf{u}(r,z)|_{S_\sigma} = u_r\mathbf{e}_r + u_\varphi\mathbf{e}_\varphi + u_z\mathbf{e}_z$$

Note that when the vector functions $\mathbf{g}(r, z)$ and $\mathbf{f}(r, z)$ have a special form, the initial axisymmetric problem can be split into the problem of the longitudinal vibrations of a solid of revolution $\mathbf{u} = u_r(r, z)\mathbf{e}_r + u_z(r, z)\mathbf{e}_z$ and torsional vibrations $\mathbf{u} = u_\varphi(r, z)\mathbf{e}_\varphi$.

3. THE TORSION PROBLEM

We will consider the axially symmetric problem of the steady torsional vibrations of an isotropic elastic beam with a variable cross-section and an axis of symmetry which coincides with the z axis. We will assume that

$$\mathbf{f} = f(r, z)\mathbf{e}_\varphi, \quad \mathbf{g} = g(r, z)\mathbf{e}_\varphi$$

and we will seek a solution in the form

$$\mathbf{u}(r, z) = u_\varphi(r, z)\mathbf{e}_\varphi$$

This problem is a special case of the problem considered in Section 2.

The difference between the torsion problem and the general case is the fact that, in the axially symmetric problem of the torsion of a beam, the structure of the Cauchy stress tensor and the structure of the displacement vector are such that the first two equations of the system of boundary integral equations (2.1) are satisfied identically. Hence, the torsion problem reduces to the boundary integral equation

$$\int_{L_u} \mathbf{K}^\sigma(r,z,\eta_1,\eta_3)\cdot \mathbf{t}_n(r,z)dL + \int_{L_\sigma} \mathbf{K}^u(r,z,\eta_1,\eta_3)\cdot \mathbf{u}(r,z)dL = \mathbf{F}(r,z) \tag{3.1}$$

where

$$-\mathbf{F}(r,z) = \int_{L_\sigma} \mathbf{K}^\sigma(r,z,\eta_1,\eta_3)\cdot \mathbf{g}(r,z)dL + \int_{L_u} \mathbf{K}^u(r,z,\eta_1,\eta_3)\cdot \mathbf{f}(r,z)dL$$

$$\eta_1^2 + \eta_3^2 = 1, \quad \mathbf{K}^\sigma = i\eta_3 G_{12}\mathbf{e}_\varphi, \quad \mathbf{K}^u = (C_1G_{22} + C_2G_{12})\mathbf{e}_\varphi$$

4. DISCRETIZATION OF THE PROBLEM IN THE AXISYMMETRIC CASE

To solve the system of boundary integral equations, obtained in Section 3, we will use the ideas of the classical method of boundary integral equations [12]. To do this we split the surface S into elements S_q, each of which is part of a surface of revolution enclosed between two parallel planes, perpendicular to the axis of symmetry of the solid of revolution. According to the boundary-element method, the displacements and forces are specified in the form of piecewise-interpolating functions (assumed constant in the simplest case) on each of the elements S_q.

We will consider the torsion problem and the corresponding boundary integral equation (3.1). We introduce the partitions

$$L_u = \sum_{q=1}^{N} L_{uq}, \quad L_\sigma = \sum_{q=1}^{M} L_{\sigma q} \tag{4.1}$$

and we use a piecewise-constant approximation of the initial functions on each element. We introduce the notation

$$t_{n\varphi}|_{L_{uq}} = X_q, \quad q = 1,\dots,N; \quad u_\varphi|_{L_{\sigma q}} = Y_q, \quad q = 1,\dots,M \tag{4.2}$$

Suppose $\{\eta_p\}_{p=1}^{P}$ is a certain set of collocation points (everywhere henceforth $p = 1, \dots, P$).

We will require that boundary integral equation (3.1) is satisfied when η runs through the set of collocation points. Taking the partitions (4.1) and approximation (4.2) into account, we obtain

$$\sum_{q=1}^{N} A_{pq}X_q + \sum_{q=1}^{M} B_{pq}Y_q = F_p \tag{4.3}$$

$$A_{pq} = \int_{L_{uq}} K_3^\sigma(\eta_p)dL, \quad q = 1,\dots,N; \quad B_{pq} = \int_{L_{\sigma q}} K_3^u(\eta_p)dL, \quad q = 1,\dots,M$$

$$-F_p = \sum_{q=1}^{N} f_q A_{pq} + \sum_{q=1}^{M} g_q B_{pq}$$

where f_q and g_q are the averaged values of the functions f and g on the elements L_{uq}, $L_{\sigma q}$.

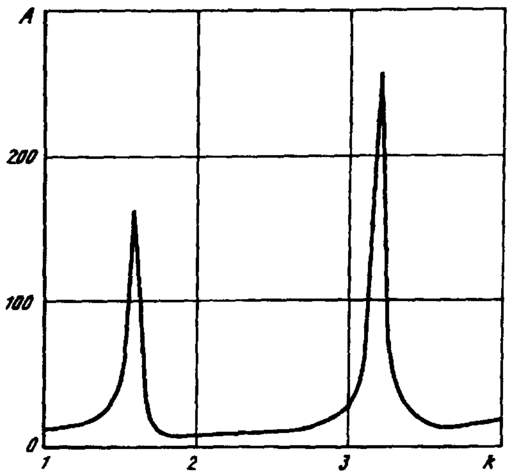

Fig. 1.

To calculate F_p one can also use the Gauss quadrature formulae, but it is this method of calculating F_p in terms of the nodal values of f_q and g_q that enables one to obtain a discrete operator in which the orders of the approximation errors of the left- and right-hand sides are the same. This is essential when inverting a completely continuous operator [9].

Thus, system (4.3) is a discrete analogue of boundary integral equation (3.1).

5. NUMERICAL EXAMPLE

We will consider, as an example, the problem of the steady torsional vibrations of an isotropic elastic cylindrical beam of height $2l$ and radius a. Suppose the lower base $z = -l$ is rigidly clamped while the displacements $\mathbf{u} = pr\mathbf{e}_\varphi$ are specified on the upper base $z = l$, and the side surface is stress-free $\mathbf{t}_n = 0$. The discrete analogue of the boundary integral equation for this torsion problem was presented in Section 4. Numerical experiments showed that, to obtain the maximum effective results, one needs to separate the imaginary and real parts of Eq. (4.3), and then each of these systems must be satisfied, for example, when $0 \leqslant \psi \leqslant \pi$ and $\pi < \psi < 2\pi$, thereby forming a square matrix of the equations of these systems. This algorithm for setting up the system enables one to avoid the linear dependence of the equations, which arises when only the real part or the imaginary part of the system is used, when $0 \leqslant \psi \leqslant 2\pi$.

However, the matrix of the discrete operator (4.3) is ill-conditioned, which is a direct consequence of the fact that the procedure for inverting a completely continuous operator is ill-posed. Hence, to find the required boundary values from Eq. (4.3) one needs to use special algorithms, for example, Tikhonov's method or the Paige–Saunders algorithm [13], used in this paper. The solution obtained differs from the exact solution by less than 1% when $M + N = 30$.

To determine the resonance frequencies we introduce the amplitude function $A = \max_q |X_q|$, which takes the maximum value from all the values of the required unknowns, obtained on the boundary. A considerable increase in the amplitude function is observed in the neighbourhood of the resonance frequencies. A graph of the amplitude function in the neighbourhood of the first and second resonance frequencies for $l/a = 1$ is shown in the Fig. 1.

Table 1 presents exact frequencies k^* and calculated frequencies k for $l/a = 1$. Note that when $M + N = 30$ all the significant digits of the first four resonance frequencies k_n and k_n^* given in the table are identical.

Table 1

n	$\lvert k_n - k_n^* \rvert \times 10^5$		k_n^*
	$N = 9$	$N = 15$	
1	102	33	1.57080
2	54	21	3.14159
3	38	12	4.71239
4	177	61	6.28319

It is interesting to note that if we consider only the real or imaginary parts of the boundary integral equation, this algorithm enables one to determine either even or odd resonance frequencies (the number of the frequencies). However, this does not mean that it is inadmissible to use the complex or imaginary parts of the boundary integral equation separately. As practical experiments show, if the required functions are odd (even), the solution obtained using only the imaginary (real) part approximates quite well to the exact solution.

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